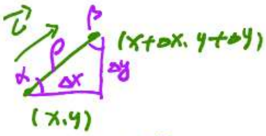
25考研 | 题型通法17笔记

题型考点　场论公式

1.方向导数

（1）定义：$\dfrac{\partial f}{\partial \vec{l}}=\lim\limits_{\rho\to0^{+}}\dfrac{f(x+\overset{\Delta x}{\boxed{\rho\cos\alpha}},y+\overset{\Delta y}{\boxed{\rho\cos\beta}})-f(x,y)}{\rho}$.



$\cos\alpha=\dfrac{\Delta x}{\rho}$ $\therefore \Delta x=\rho\cos\alpha$

$\cos\beta=\dfrac{\Delta y}{\rho}$ $\therefore \Delta y=\rho\cos\beta$

$\vec{l}$ 单位化 $(\cos\alpha,\cos\beta)$

（2）可微函数：$\dfrac{\partial f}{\partial \vec{l}}=f_x'\cos\alpha+f_y'\cos\beta$ $=(f_x',f_y')\cdot(\cos\alpha,\cos\beta)$

【试题 223】（96-1） 函数 $u=\ln\left(x+\sqrt{y^2+z^2}\right)$ 在点 $A(1,0,1)$ 处沿点 A 指向点 $B(3,-2,2)$ 方向的方向导数为______.

法1:

解: ① $u_x'=\dfrac{1}{x+\sqrt{y^2+z^2}}$, $u_x'|_A=\dfrac{1}{2}$

$u_y'=\dfrac{1}{x+\sqrt{y^2+z^2}}\cdot\dfrac{y}{\sqrt{y^2+z^2}}$, $u_y'|_A=\dfrac{1}{2}\cdot0=0$

$u_z'=\dfrac{1}{x+\sqrt{y^2+z^2}}\cdot\dfrac{z}{\sqrt{y^2+z^2}}$, $u_z'|_A=\dfrac{1}{2}\cdot1=\dfrac{1}{2}$

② $\vec{l}=\overrightarrow{AB}=(2,-2,1)$ ∴ 单位化得: $\dfrac{1}{3}(2,-2,1)$

③ $\dfrac{\partial u}{\partial \vec{l}}\Big|_A=\left(\dfrac{1}{2},0,\dfrac{1}{2}\right)\cdot\dfrac{1}{3}(2,-2,1)=\dfrac{1}{2}$

法2: $\dfrac{\partial u}{\partial \vec{l}}\Big|_A=\lim\limits_{\rho\to0^+}\dfrac{f(1+\rho\cos\alpha,\ 0+\rho\cos\beta,\ 1+\rho\cos\gamma)-f(1,0,1)}{\rho}$

$f(x,y,z)=\ln(x+\sqrt{y^2+z^2})$ $=\lim\limits_{\rho\to0^+}\dfrac{f(1+\frac{2}{3}\rho,\ -\frac{2}{3}\rho,\ 1+\frac{1}{3}\rho)-f(1,0,1)}{\rho}$

$=\lim\limits_{\rho\to0^+}\dfrac{\ln\left[1+\frac{2}{3}\rho+\sqrt{(-\frac{2}{3}\rho)^2+(1+\frac{1}{3}\rho)^2}\right]-\ln2}{\rho}$

$\overset{洛}{=}\lim\limits_{\rho\to0^+}\dfrac{1}{1+\frac{2}{3}\rho+\sqrt{(-\frac{2}{3}\rho)^2+(1+\frac{1}{3}\rho)^2}}\cdot\left(\dfrac{2}{3}+\dfrac{(\frac{2}{3}\rho)\cdot\frac{2}{3}+(1+\frac{1}{3}\rho)\cdot\frac{1}{3}}{\sqrt{(\frac{2}{3}\rho)^2+(1+\frac{1}{3}\rho)^2}}\right)$

$=\dfrac{1}{2}\cdot\left(\dfrac{2}{3}+\dfrac{1}{3}\right)=\dfrac{1}{2}$.

2.梯度　$\dfrac{\partial f}{\partial \vec{l}}=(f_x',f_y')\cdot(\cos\alpha,\cos\beta)=\|f_x',f_y'\|\cdot1\cdot\cos\theta$

（1） $\mathbf{grad}f(x,y)=\left(f'_x,f'_y\right)$.　　l沿 $\mathrm{grad}f \Leftarrow \theta=0$时. $\frac{\partial f}{\partial l}$ 最大，增加最快

l沿 $-\mathrm{grad}f \Leftarrow \theta=\pi$时. $\frac{\partial f}{\partial l}$ 最小，减小最快.

（2） $f(x,y)$ 沿 $\mathbf{grad}f(x,y)$ 的方向导数最大，最大值为 $\left\|\mathbf{grad}f(x,y)\right\|$；沿 $-\mathbf{grad}f(x,y)$ 的方向导数最小，最小值为 $-\left\|\mathbf{grad}f(x,y)\right\|$.

$l \perp \mathrm{grad}f$.　$\theta=\frac{\pi}{2}$时. $\frac{\partial f}{\partial l}=0$.

【试题 224】 (12-1.11) $grad\left(xy+\dfrac{z}{y}\right)\Big|_{(2,1,1)}=$ $\underline{(1,1,1)}$ 或 $\vec{i}+\vec{j}+\vec{k}$

$f(x,y,z)$

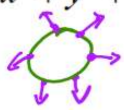
【难度】 本题的数学一难度值为 0.573.

$f_x'=y$ ， $f_x'(2,1,1)=1$

$f_y'=x-\dfrac{z}{y^2}$ ， $f_y'(2,1,1)=1$

$f_z'=\dfrac{1}{y}$ ， $f_z'(2,1,1)=1$

【试题 225】(15-1.17) 已知函数 $f(x,y)=x+y+xy$，曲线 C：$x^2+y^2+xy=3$，求 $f(x,y)$ 在曲线 C 的最大方向导数.



解：$f(x,y)$ 在 (x,y) 处的最大方向导数为

$$g(x,y)=\|grad\, f(x,y)\|=\sqrt{f_x'^2+f_y'^2}=\sqrt{(1+y)^2+(1+x)^2}$$

令 $L(x,y,\lambda)=(1+y)^2+(1+x)^2+\lambda(x^2+y^2+xy-3)$，则

$$\begin{cases} L_x'=2(1+x)+\lambda(2x+y)=0 & ① \\ L_y'=2(1+y)+\lambda(2y+x)=0 & ② \\ L_\lambda'=x^2+y^2+xy-3=0 \end{cases}$$

① - ② 得：

$2(x-y)+\lambda(x-y)=0$

$\therefore (x-y)(\lambda+2)=0$

$\therefore y=x$ 或 $\lambda=-2$

解得：$\begin{cases}x=1\\y=1\end{cases}$ 或 $\begin{cases}x=-1\\y=-1\end{cases}$

$\begin{cases}x=2\\y=-1\end{cases}$ 或 $\begin{cases}x=-1\\y=2\end{cases}$

$g(1,1)=2\sqrt{2}$， $g(-1,-1)=0$， $g(2,-1)=g(-1,2)=3$

$\therefore f(x,y)$ 在曲线上的最大方向导数为：3.

3.散度：设向量场 $\boldsymbol{A}(x,y,z)=P(x,y,z)\boldsymbol{i}+Q(x,y,z)\boldsymbol{j}+R(x,y,z)\boldsymbol{k}$，则

标量.

$$\boldsymbol{div\,A}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}.$$

4.旋度：设向量场 $\boldsymbol{A}(x,y,z)=P(x,y,z)\boldsymbol{i}+Q(x,y,z)\boldsymbol{j}+R(x,y,z)\boldsymbol{k}$，则

矢量

$$\boldsymbol{rot\,A}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

【试题 226】（16-1.10） 向量场 $\boldsymbol{A}(x,y,z)=(x+y+z)\boldsymbol{i}+xy\boldsymbol{j}+z\boldsymbol{k}$ 的旋度

$\mathbf{rot}\,\boldsymbol{A}=$ ______.

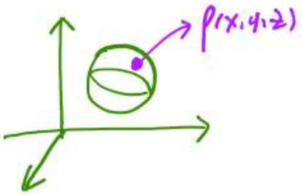
$$\operatorname{rot}\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y+z & xy & z\end{vmatrix}=(0,1,y-1)\ ✓$$

或 $\vec{j}+(y-1)\vec{k}$. ✓

第九章 多元函数积分学01（仅数一）

题型考点 三重积分的计算

一、概念



$m = \iiint_\Omega \rho(x,y,z)\,dv$

$= \rho(\xi,\eta,\gamma)\cdot V,\ (\xi,\eta,\gamma)\in\Omega.$

二、性质

1.普通对称性

(1) Ω关于 yoz 面对称，x 坐标具有对称性. $\iiint_\Omega f(x,y,z)dv$ 关于x偶倍奇0.

(2) xoz y y

(3) xoy z z

2.轮换对称性

$\Omega_{x,y,z} \xrightarrow[\substack{x\to y\\ y\to z\\ z\to x}]{\text{轮换}} \Omega_{yzx} = \Omega_{xyz}$

例: ① $\Omega=\{(x,y,z)\mid x^2+y^2+z^2\le 1\}$ 轮换、对换.

② $\Omega=\{(x,y,z)\mid x^2+y^2+z^2\le 1,\ z\ge 0\}$ 不能轮换. x,y 可对换.

$I = \iiint_\Omega f(x,y,z)\,dv = \iiint_\Omega f(y,z,x)\,dv = \iiint_\Omega f(z,x,y)\,dv$

$= \frac{1}{3}\iiint_\Omega [f(x,y,z)+f(y,z,x)+f(z,x,y)]\,dv$

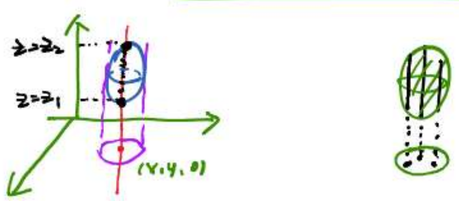
$I = \iiint_{\Omega_{xyz}} f(x,y,z)\,dv = \iiint_{\Omega_{yzx}} f(y,z,x)\,dv$ ✓，但不能化简.

三、计算

1.投影法

$$\iiint_\Omega f(x,y,z)dv = \iint_{D投}\left(dxdy\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz\right)$$

→ (x,y,0)点对应"丝"质量

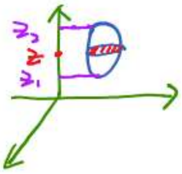


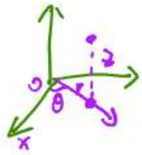
2.截面法

$$\iiint\limits_{\Omega} f(x,y,z)dv=\int_{z_1}^{z_2}\left(dz\iint\limits_{D截} f(x,y,z)dxdy\right)$$

与z有关

z处对应"片"的质量.



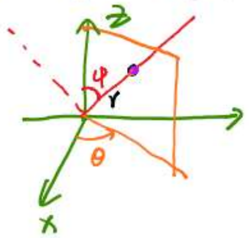


3.柱面坐标＝投影法＋极坐标、

$$\iiint\limits_{\Omega} f(x,y,z)dv=\int_{\alpha}^{\beta}d\theta\int_{r_1(\theta)}^{r_2(\theta)}rdr\int_{z_1(\theta,r)}^{z_2(\theta,r)}f(r\cos\theta,r\sin\theta,z)dz$$

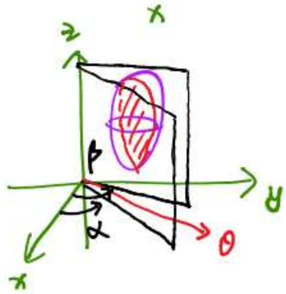
4.球面坐标

$$\iiint_{\Omega} f(x,y,z)dv=\int_{\alpha}^{\beta}d\theta\int_{\varphi_1(\theta)}^{\varphi_2(\theta)}d\varphi\int_{r_1(\theta,\varphi)}^{r_2(\theta,\varphi)}f(r\sin\varphi\cos\theta,r\sin\varphi\sin\theta,r\cos\varphi)r^2\sin\varphi dr$$



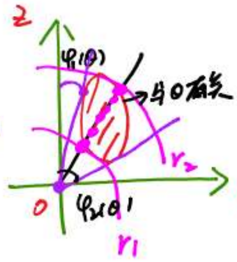
$$\begin{cases} x=r\sin\varphi\cos\theta \\ y=r\sin\varphi\sin\theta \\ z=r\cos\varphi \end{cases},\quad dv=r^2\sin\varphi\, dr\, d\varphi\, d\theta$$

①



$\alpha\le\theta\le\beta$ →

②

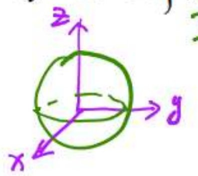


与θ有关 $\varphi_1(\theta)\le\varphi\le\varphi_2(\theta)$

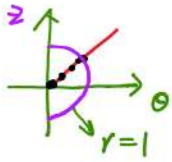
【试题 227】(09-1.12) 设 $\Omega=\{(x,y,z)|x^2+y^2+z^2\le1\}$，则 $I=\iiint\limits_{\Omega} z^2dxdydz=$______.

【难度】 数学一难度值为 0.346.

① Ω



→



② 对称性

$$I=\iiint_{\Omega}x^2dv=\iiint_{\Omega}y^2dv$$

$$=\frac{1}{3}\iiint_{\Omega}(x^2+y^2+z^2)\,dv$$

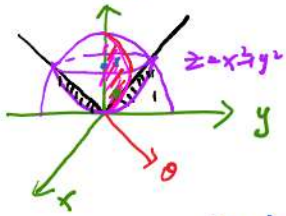
$$=\frac{1}{3}\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^1 r^2\cdot r^2\sin\varphi\, dr$$

$$=\frac{1}{3}\cdot2\pi\cdot\int_0^{\pi}\sin\varphi\, d\varphi\cdot\frac{1}{5}r^5\Big|_0^1$$

$$=\frac{2}{3}\pi\cdot2\cdot\frac{1}{5}=\frac{4}{15}\pi.$$

例：$\Omega=\{(x,y,z)\,|\,,\ x^2+y^2\le z\le\sqrt{2-x^2-y^2}\}$

计算 $I=\iiint\limits_{\Omega}dv$



解：$I=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_0^{\sqrt{2}}r^2\sin\varphi\, dr$

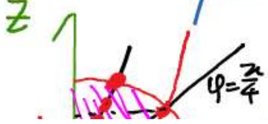
$+\int_0^{2\pi}d\theta\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\varphi\int_0^{\frac{\cos\varphi}{\sin^2\varphi}}r^2\sin\varphi\, dr$

$+\int_0^{2\pi}d\theta\int_0^1 r\,dr\int_{r^2}^{r}dz$

$z=x^2+y^2$

$\therefore r\cos\varphi=r^2\sin^2\varphi$

$\therefore r=\frac{\cos\varphi}{\sin^2\varphi}$

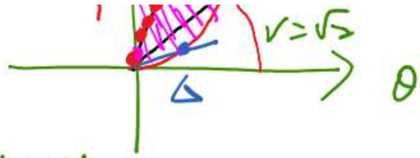


$$= 2\pi \cdot (-\cos\varphi)\Big|_0^{\frac{\pi}{4}} \cdot \frac{1}{3}r^3\Big|_0^{\sqrt{2}}$$

$$+ 2\pi \cdot \int_0^1 r(r-r^2)\,dr$$

$$= 2\pi\left(1-\frac{\sqrt{2}}{2}\right)\cdot\frac{2}{3}\sqrt{2} + 2\pi\left(\frac{1}{3}r^3-\frac{1}{4}r^4\right)\Big|_0^1$$

$$= \left(\frac{4}{3}\sqrt{2}-\frac{7}{6}\right)\pi.$$



四、应用

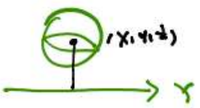
1.质心、形心

（1）$\bar{x}=\dfrac{\iiint\limits_{\Omega} xf(x,y,z)dV}{\iiint\limits_{\Omega} f(x,y,z)dV}$

（2）$\bar{x}=\dfrac{\iiint\limits_{\Omega} xdV}{\iiint\limits_{\Omega} dV}$

2.转动惯量

$$I=\frac{1}{2}mV^2=\frac{1}{2}m(r\omega)^2=\frac{1}{2}\overset{I}{\overset{\|}{\boxed{mr^2}}}\omega^2$$

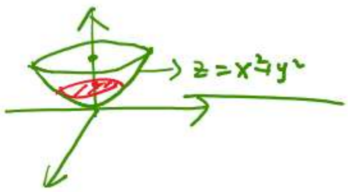


$$I=\iiint\limits_{\Omega}(y^2+z^2)\cdot\rho(x,y,z)\,dv$$

【试题 228】（10-1.12） 设$\Omega=\left\{(x,y,z)\middle|x^2+y^2\le z\le 1\right\}$，则$\Omega$的形心的竖坐标

$\bar{z}=$________.

【难度】 本题的数学一难度值为 0.295.



$$① \iiint\limits_{\Omega} z\,dv=\int_0^1 dz\iint\limits_{x^2+y^2\le z} z\,dxdy$$

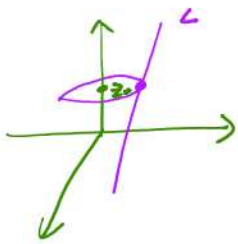
$$=\int_0^1 z\cdot\pi z\,dz=\frac{\pi}{3}z^3\Big|_0^1=\frac{\pi}{3}$$

$$② \iiint\limits_{\Omega} 1\,dv=\int_0^1 dz\iint\limits_{x^2+y^2\le z} 1\,dxdy=\int_0^1 \pi z\,dz=\frac{\pi}{2}z^2\Big|_0^1=\frac{\pi}{2}$$

$$\bar{z}=\frac{\iiint z\,dv}{\iiint 1\,dv}=\frac{\frac{\pi}{3}}{\frac{\pi}{2}}=\frac{2}{3}.$$

【试题 229】（13-1.19） 设直线L过$A(1,0,0)$，$B(0,1,1)$两点，将L绕z轴旋转一周得到曲面Σ，Σ与平面$z=0$，$z=2$所围成的立体为Ω.

（Ⅰ）求曲面Σ的方程；（Ⅱ）求Ω的形心坐标.



【难度】 本题的数学一难度值为 0.289.

解：(1) $\vec{AB}=(-1,1,1)$ ∴ L的方程为：$\dfrac{x-1}{-1}=\dfrac{y}{1}=\dfrac{z}{1}$

解：(1)

$\forall (x_0, y_0, z_0) \in L$，则：$1-x_0 = y_0 = z_0$

线绕 z 轴旋转的轨迹为：$\begin{cases} x^2+y^2 = x_0^2+y_0^2 \\ z = z_0 \end{cases}$

消去 x_0, y_0, z_0 得 Σ 的方程：

$$x^2+y^2 = (1-z)^2 + z^2 = 1-2z+2z^2$$

(2) 由对称性可知：$\bar{x} = \bar{y} = 0$

$$\iiint_{\Omega} z \, dV = \int_0^2 dz \iint_{x^2+y^2 \le 1-2z+2z^2} z \, dxdy$$

$$= \int_0^2 \pi(1-2z+2z^2) \cdot z \, dz$$

$$= \pi\left(\frac{1}{2}z^2 - \frac{2}{3}z^3 + \frac{1}{2}z^4\right)\Big|_0^2 = \frac{14}{3}\pi.$$

$$\iiint_{\Omega} 1 \, dV = \int_0^2 dz \iint_{x^2+y^2 \le 1-2z+2z^2} dxdy$$

$$= \int_0^2 \pi(1-2z+2z^2) \, dz$$

$$= \pi\left(z - z^2 + \frac{2}{3}z^3\right)\Big|_0^2 = \frac{10}{3}\pi$$

$\therefore \bar{z} = \dfrac{\frac{14}{3}\pi}{\frac{10}{3}\pi} = \dfrac{7}{5}$ $\quad\therefore$ 形心坐标为 $(0, 0, \frac{7}{5})$