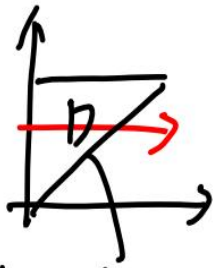
1.计算二重积分 $\iint\limits_D \sqrt{y^2-xy}\,\mathrm{d}x\mathrm{d}y$，其中 D 是由直线 $y=x, y=1, x=0$ 所围成的平面区域.

原式 $=\int_0^1 dy\int_0^y \sqrt{y^2-xy}\,dx$



$=\int_0^1 -\frac{1}{y}\left[\frac{2}{3}(y^2-xy)^{\frac{3}{2}}\right]_0^y dy$ $x=y$

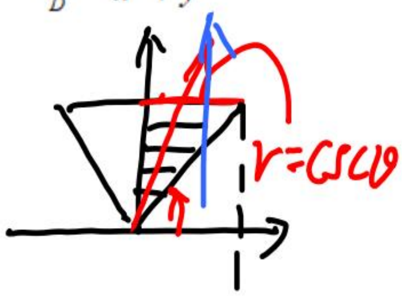
$=\int_0^1 \cancel{\frac{1}{y}}\cdot\frac{2}{3}\cdot y^{\cancel{3}\,2}dy=\frac{2}{3}\left[\frac{1}{3}y^3\right]_0^1=\frac{2}{9}.$

①对称性
②直/极.
先x 先y.
③分割区域

$r\sin\theta=1 \Rightarrow r=\frac{1}{\sin\theta}$

2. 设 D 是由直线 $y=1$, $y=x$, $y=-x$ 围成的有界区域，计算二重积分 $\iint\limits_D \frac{x^2-\cancel{xy}-y^2}{x^2+y^2}\mathrm{d}x\mathrm{d}y$.

记 $f(x,y)=\frac{xy}{x^2+y^2}$



$\begin{cases} D\text{关于}y\text{轴对称} \\ f(-x,y)=-f(x,y) \end{cases} \Rightarrow \iint\limits_D f(x,y)\,d\sigma=0.$

记 $g(x,y)=\frac{x^2-y^2}{x^2+y^2}$ $\begin{cases} D\text{关于}y\text{轴对称} \\ g(-x,y)=g(x,y) \end{cases} \Rightarrow \iint\limits_D g(x,y)\,d\sigma$

$=2\iint\limits_{D_{右}} g(x,y)\,d\sigma$

原式 $=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_0^{\csc\theta}\frac{\cancel{r^2}(\cos^2\theta-\sin^2\theta)}{\cancel{r^2}}r\,dr$

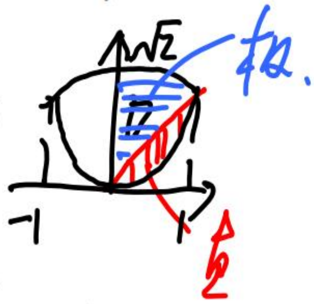
$\csc^2\theta-\cot^2\theta=1$

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\csc^2\theta(\cos^2\theta-\sin^2\theta)\,d\theta=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\cot^2\theta-1)\,d\theta$

$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\csc^2\theta-2)\,d\theta=\left[-\cot\theta-2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=1-\frac{\pi}{2}.$

3. 计算二重积分 $\iint\limits_D x(x+\cancel{y})\mathrm{d}x\mathrm{d}y$，其中 $D=\left\{(x,y)\mid x^2+y^2\le 2, y\ge x^2\right\}$.

原式 $=2\iint\limits_{D_1}\underline{x^2}\,dxdy$ （D_1为D在y轴右侧部分）



$$=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_0^{\sqrt{2}}r^3\cos^2\theta\,dr+2\int_0^1dx\int_{x^2}^{x}x^2dy$$

$\frac{1+\cos2\theta}{2}$

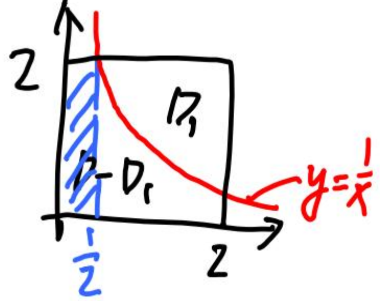
$$=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\underline{\cos^2\theta}\,d\theta\int_0^{\sqrt{2}}r^3dr+2\int_0^1x^2(x-x^2)dx$$

$$=\left[\theta+\frac{1}{2}\sin2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cdot\cancel{\left[\frac{r^4}{4}\right]_0^{\sqrt{2}}}+2\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_0^1$$

$$=\frac{\pi}{2}-\frac{\pi}{4}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\frac{2}{5}=\frac{\pi}{4}-\frac{2}{5}.$$

4 计算 $\iint\limits_D\max\{xy,1\}\mathrm{d}x\mathrm{d}y$，其中 $D=\{(x,y)|0\le x\le 2,0\le y\le 2\}$.

$$\max\{xy,1\}=\begin{cases}xy, & y\ge\frac{1}{x}\\ 1, & y<\frac{1}{x}\end{cases}$$



记D_1为D在$y=\frac{1}{x}$上方的部分.

$$\text{原式}=\iint\limits_{D_1}xy\,dxdy+\iint\limits_{D-D_1}dxdy$$

$$=\int_{\frac{1}{2}}^{2}dx\int_{\frac{1}{x}}^{2}xy\,dy+2\times\frac{1}{2}+\int_{\frac{1}{2}}^{2}\frac{1}{x}dx$$

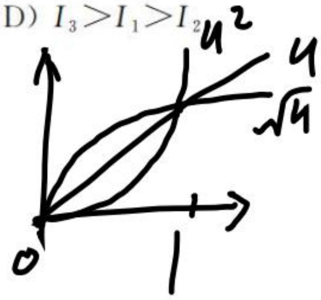
$$=\int_{\frac{1}{2}}^{2}\frac{x}{2}\left(4-\frac{1}{x^2}\right)dx+1+\left[\ln|x|\right]_{\frac{1}{2}}^{2}$$

$$=\frac{1}{2}\left[2\underline{x^2}-\ln|x|\right]_{\frac{1}{2}}^{2}+\underline{\left[\ln|x|\right]_{\frac{1}{2}}^{2}}+1=5-\frac{1}{4}+\frac{1}{2}\left(\ln2-\ln\frac{1}{2}\right)$$

$$=\frac{19}{4}+\ln2$$

3. 设 $I_1=\iint\limits_D \cos\sqrt{x^2+y^2}\,d\sigma$，$I_2=\iint\limits_D \cos(x^2+y^2)\,d\sigma$，$I_3=\iint\limits_D \cos(x^2+y^2)^2\,d\sigma$，其中 $D=\{(x,y)\mid x^2+y^2\leqslant 1\}$，则（ A ）

(A) $I_3>I_2>I_1$.　　(B) $I_1>I_2>I_3$.　　(C) $I_2>I_1>I_3$.　　(D) $I_3>I_1>I_2$.



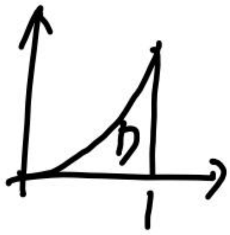
比较当 $0\le u\le 1$ 时，$\sqrt{u}$，u，u^2 的大小

$\cos x$ 在 $[0,1]$ ↓

4. 设 $f(x,y)$ 连续，且 $f(x,y)=xy+\iint\limits_D f(u,v)\,du\,dv$，其中 D 是由 $y=0, y=x^2, x=1$ 所围成的区域，则 $f(x,y)$ 等于（ C ）

(A) xy.　　(B) $2xy$.　　(C) $xy+\frac{1}{8}$.　　(D) $xy+1$.

设 $\iint\limits_D f(u,v)\,du\,dv=a$，则 $f(x,y)=xy+a$



$$a=\int_0^1 dx\int_0^{x^2}(xy+a)\,dy$$

$$=\int_0^1\left[x\frac{y^2}{2}+ay\right]_0^{x^2}dx$$

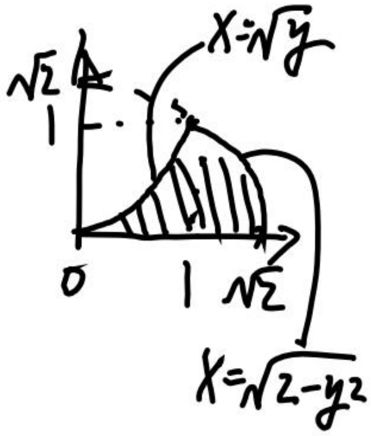
$$=\int_0^1\left(\frac{1}{2}x^5+ax^2\right)dx=\left[\frac{1}{12}x^6+\frac{a}{3}x^3\right]_0^1=\frac{1}{12}+\frac{a}{3}$$

$$\Rightarrow 3a=\frac{1}{4}+a\Rightarrow a=\frac{1}{8}$$

7. 交换积分次序：$\int_0^1 dy\int_{\sqrt{y}}^{\sqrt{2-y^2}} f(x,y)\,dx=$ ________.

$x=\sqrt{y}\Rightarrow y=x^2$

$x=\sqrt{2-y^2}\Rightarrow x^2+y^2=2$

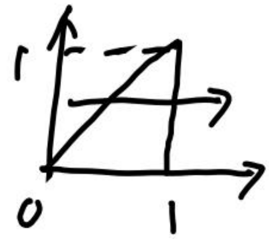


$$\text{原式}=\int_0^1 dx\int_0^{x^2} f(x,y)\,dy$$

$$+\int_1^{\sqrt{2}} dx\int_0^{\sqrt{2-x^2}} f(x,y)\,dy.$$

8. $\int_0^1 \mathrm{d}x\int_0^x x\sqrt{1-x^2+y^2}\,\mathrm{d}y =$ ____________.

原式 $=\int_0^1 dy\int_y^1 x\sqrt{1-x^2+y^2}\,dx$

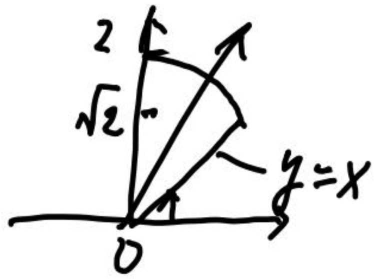


$=\int_0^1 -\frac{1}{2}\cdot\frac{2}{3}\left[(1-x^2+y^2)^{\frac{3}{2}}\right]_y^1 dy$

$=-\frac{1}{3}\int_0^1 (y^3-1)dy=-\frac{1}{3}\left[\frac{y^4}{4}-y\right]_0^1=\frac{1}{4}.$

9. $\int_0^{\sqrt{2}} \mathrm{e}^{-y^2}\mathrm{d}y\int_0^y \mathrm{e}^{-x^2}\mathrm{d}x+\int_{\sqrt{2}}^2 \mathrm{e}^{-y^2}\mathrm{d}y\int_0^{\sqrt{4-y^2}} \mathrm{e}^{-x^2}\mathrm{d}x =$ ____________.

原式 $=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta\int_0^2 e^{-r^2} r\,dr$



$=\frac{\pi}{4}\left[-\frac{1}{2}e^{-r^2}\right]_0^2=\frac{\pi}{8}(1-e^{-4})$

10. 设 $f(x)$ 是定义在 $[0,1]$ 上的连续函数，且 $f(0)=1$，则 $\lim\limits_{x\to0^+}\dfrac{\int_0^{x^2}\mathrm{d}t\int_{\sqrt{t}}^{x} f(u)\mathrm{d}u}{x(\cos x-1)}=$

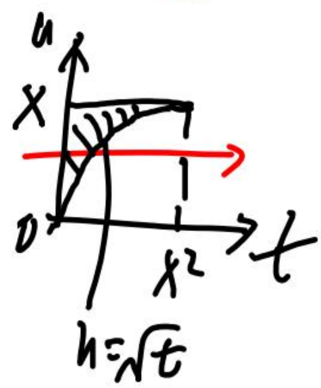
____________.

$\sim -\frac{1}{2}x^2$

$\int_0^{x^2}\left(\int_{\sqrt{t}}^{x} f(u)du\right)dt=\int_0^x du\int_0^{u^2} f(u)dt$

$g(x,t)$

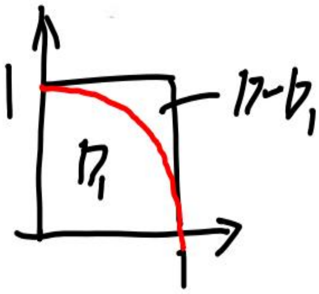
$=\int_0^x u^2 f(u)du.$



原式 $=\lim\limits_{x\to0^+}\dfrac{\int_0^x u^2 f(u)du}{-\frac{1}{2}x^3}\xlongequal[\text{洛}]{\frac{0}{0}}\lim\limits_{x\to0^+}\dfrac{x^2 f(x)}{-\frac{3}{2}x^2}=-\frac{2}{3}f(0)=-\frac{2}{3}.$

19. 计算二重积分$\iint\limits_{D}|x^2+y^2-1|\,d\sigma$，其中$D=\{(x,y)\,|\,0\leqslant x\leqslant 1,0\leqslant y\leqslant 1\}$.

记 $D_1=\{(x,y)\,|\,x\geq 0,y\geq 0,x^2+y^2\leq 1\}$



原式 $=\iint\limits_{D_1}(1-x^2-y^2)\,d\sigma+\iint\limits_{D-D_1}(x^2+y^2-1)\,d\sigma$

$=\iint\limits_{D_1}(1-x^2-y^2)\,d\sigma+\iint\limits_{D}(x^2+y^2-1)\,d\sigma-\iint\limits_{D_1}(x^2+y^2-1)\,d\sigma$

$=2\iint\limits_{D_1}(1-x^2-y^2)\,d\sigma+\iint\limits_{D}(x^2+y^2-1)\,d\sigma$

$=2\int_0^{\frac{\pi}{2}}d\theta\int_0^1(1-r^2)r\,dr+\int_0^1dx\int_0^1(x^2+y^2-1)\,dy$

$=\pi\left(\frac{1}{2}-\frac{1}{4}\right)+\int_0^1\left(x^2-1+\frac{1}{3}\right)dx$

$=\frac{\pi}{4}+\frac{1}{3}-\frac{2}{3}=\frac{\pi}{4}-\frac{1}{3}.$